AVERAGED CRITERION OF BINDER STRENGTH IN FIBER COMPOSITES

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This article examines the construction of a criterion of the strength of the binder in composite materials reinforced with high-modulus fibers. The criterion is expressed in terms of averaged (macroscopic) stresses. The solution is obtained on the basis of analysis of a "cellular problem" [1] using hypotheses which consider the substantial difference between the stiffnesses of the reinforcing fibers and the binder.

In the deformation of fiber composites most of the load is taken up by the high-modulus fibers [1]. The stress—strain state of the binder in this case can be determined by using an approximate model in which the soft binder "follows" the reinforcing fibers as a rigid framework which is deformed independently of the binder. An approach very close to this is widely used in studying unidirectional composites [2-4], where it leads to models that show satisfactory agreement with experimental results [5, 6].

We will examine a composite consisting of periodically alternating layers of fibers laid parallel to the plane Ox_1x_2. The fibers are arranged parallel to each other within a given layer. The angle formed by the fibers of the α-th layer with the axis Ox_1 will be designated as φ_α, $\alpha = 1, \ldots, M$ (M is the number of reinforcing layers comprising a period of the structure of the composite). The reinforcement scheme just described is typical of composites made by winding or prepeg technology [7]. We will examine averaged deformations of the form e_{ij}, $e_{i3} = 0$ $(i = 1, 2)$. i.e., strains in the plane of the reinforcing layers. These are the strains of the greatest practical importance for the type of composite being considered.

The problem of the deformation of a composite with the condition that the diameter ϵ of the reinforcing fibers is small ($\epsilon \ll 1$) can be reduced to the solution of a problem of the theory of elasticity on a cell of the periodicity Π in the structure of a composite [1, 8, 9]. Specifically, if $\{e_{ij}\}$ is the averaged strain tensor of the composite, the theory of elasticity problem should be solved with zero body forces and the boundary condition

$$\mathbf{w} - e_{ij}x_j\mathbf{e}_i,\ \boldsymbol{\sigma}_n(\mathbf{w} - e_{ij}x_j\mathbf{e}_i)\ \text{periodicity of}\ \Pi, \tag{1}$$

where $\mathbf{w}$ are displacements; $\boldsymbol{\sigma}_n(\mathbf{u})$ are the normal stresses responsible for $\mathbf{u}$.

Joining of the solutions of the elasticity theory problem with boundary condition (1) in the finite region Q gives the solution of the elastic problem for the composite as a whole with an error which approaches zero in $W_2^1(Q)$ at $\epsilon \to 0$ [1]. In the case of an infinite region Q, joining of the elastic solutions with the boundary condition gives an exact solution of a periodic problem of elasticity theory corresponding to the mean strain $\{e_{ij}\}$ [9]. Our examination is based on the hypothesis that local (microscopic) stresses $\{\sigma_{ij}^\epsilon\}$ in the composite — at least outside the edge-effect region — coincide with the stresses determined from the solution of the theory of elasticity problem with boundary condition (1).

Note. 1. The above applies to bodies of appreciable dimensions along all three axes. In the case of small plates, when the dimension of the body in the direction of the axis Ox_3 is also on the order of ϵ, formal asymptotic expansion leads to the auxiliary condition

$$\boldsymbol{\sigma}_n(\mathbf{w} - e_{ij}x_j\mathbf{e}_i) = 0 \tag{2}$$

on the edges of the cell Π perpendicular to the axis Ox_3 (condition of periodicity of $\mathbf{w} - e_{ij}x_j\mathbf{e}_i$ is omitted in this case with respect to the variable x_3).

We will study a problem of the theory of elasticity with boundary condition (1), using the above model of rigid fibers in a flexible binder. The composites being examined are characterized by a ratio of stiffnesses of the components $E_f/E \gg 1$ (E_f, E are the Young's

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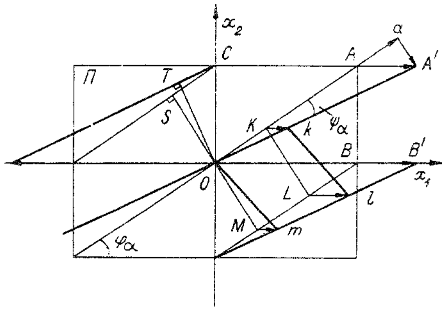

Fig. 1

moduli of the fibers and binder). If this condition is satisfied, then we have $\sigma_f/\sigma \sim E_f/E \gg 1$, where σ_f and σ are the characteristic stresses in the fibers and binder, respectively (assuming that stress concentrators are absent). In light of this, the stresses in the binder have no effect on the deformation of the fibers (more exactly, this affect is on the order of $\sigma/\sigma_f \ll 1$, and we will ignore it, permitting an error on the order of $E/E_f \ll 1$). The hypothesis for the fibers follows from this.

1. The deformation of the reinforcing fibers (forming the disconnected framework) is determined independently of the binder. As regards the binder, due to the foregoing the fibers are perfectly rigid in relation to the binder (more exactly, they are "very" stiff, with a relative stiffness $E_f/E \gg 1$). There is ideal contact between the fibers and the bindar with respect to the displacements. This leads to the hypothesis for the binder.

2. The strains of the binder are determined from the solution of the problem of the deformation of the elastic material which "follows" the rigid framework with specified displacements **w** of the latter.

The above-described approach to constructing a model of rigid fibers in a flexible binder is mechanical in character. It is understood that this approach can be formalized. By introducing the parameter $E_f/E \gg 1$, we arrive at problems of the "rigid expansion" type [10] or of the type studied in [1]. The equations obtained on the basis of hypotheses 1 and 2 and the methods in [1, 10] coincide.

If boundary condition (1) is assigned, then the displacements of the reinforcing fibers lie in planes parallel to Ox_1x_2. In conformity with the structure of the composite (alternating layers of parallel-arranged fibers), the displacements are conveniently represented as the sum of the mutual displacements of the fibers within a reinforcing layer and the displacements of the layers themselves.

<u>Interfiber Strains of the Binder.</u> We will examine the deformation of the components of a composite within the layer of reinforcing fibers. We begin with the special case $e_{ij} = e_{11}\delta_{i1}\cdot\delta_{j1}$ $(i, j = 1,2)$, which corresponds to the averaged tensile–compressive strain along the axis Ox_1. Using hypothesis 1, we find the strains are the framework formed by the reinforcing fibers. The displacements of the fibers are shown in Fig. 1, where the light and dark lines represent the axes of the fibers before and after deformation. The displacements AA' and BB' are equal to $e_{11}|OB|$, where $2|OB|$ is the size of a cell of the periodicity Π along the axis Ox_1. It follows from Fig. 1 that the fibers remain parallel in the case of deformation. Now, using hypothesis 2, we examine the deformation of the binder. We isolate the binder element OKLM in Fig. 1. By virtue of hypothesis 2, the deformation of the binder – including the element OKLM – consists of its "following" the stiff fibers. For the element OKLM, this means that it becomes the new element Okℓm. Here, the element has undergone deformation which can be represented as the sum of the tensile–compressive strain $e^{\varepsilon}_{\ell\ell}$ direction of the fiber axis, the shear strains $e^{\varepsilon}_{ln}, e^{\varepsilon}_{nl}$, and the tensile–compressive strain e^{ε}_{nn} in the direction perpendicular to the fiber axis. Here and below, we use a local coordinate system in which the axis ℓ coincides with the fiber direction in the reinforcing layer and the axis n is perpendicular to ℓ. The tensor $e_{ij}(i, j = \ell, n)$ is transformed by the standard method in changing over to the coordinate system Ox_1x_2.

Let us calculate these strains of a binder element.

1. The deformation of the reinforcing fibers in the direction of their axis (see Fig. 1)

$$e^{\varepsilon}_{ll} = \frac{|Aa|}{|OA|} = e_{11}\cos^2\varphi_\alpha.$$

"Following" the rigid framework, the binder element OKLM undergoes the same deformation in the fiber direction.

2. Since, as noted above, fibers remain parallel to each other during the deformation of the framework, then $e^{\varepsilon}_{ln} = e^{\varepsilon}_{nl} = \psi_\alpha$ (angle ψ_α is indicated in Fig. 1). We have $\operatorname{tg}\psi_\alpha = \frac{|A'a|}{|OA| + |Aa|}$. Since we are interested in the linear (with respect to e_{11}). Part of the strain and its corresponding small angle ψ_α, then by a standard method we find from the last equality that $\psi_\alpha \approx \operatorname{tg}\psi_\alpha = \frac{|A'a|}{|OA| + |Aa|} \approx \frac{|A'a|}{|OA|} = e_{11}\sin\varphi_\alpha\cos\varphi_\alpha$ (here, we made use of the fact that |A'a| and |Aa| have the order of e_{11}).

3. The presence of nontrivial deformation of the binder e^{ϵ}_{nn} is due to the fact that in the deformation of a cell of the periodicity Π, the adjacent reinforcing fibers come closer to each other (see Fig. 1). We will calculate the distances |OS| and |OT| between adjacent fibers before and after deformation. We will examine ΔOSC and ΔOTC, having the common hypotenuse OC. It is evident from Fig. 1 that $|OS| = |OC|\cos\varphi_\alpha$, while $|OT| = |OC|\cos(\varphi_\alpha - \psi_\alpha)$. Then $|OT| - |OS| \approx \psi_\alpha|OC|\sin\varphi_\alpha$. Here, in the expansion $\cos(\varphi_\alpha - \psi_\alpha)$, we retain a term which is linear with respect to ψ_α and, thus, to e_{11}. Then $e^{\varepsilon}_{nn} = \frac{\psi_\alpha|OC|\sin\varphi_\alpha}{|OS|} = e_{11}\sin^2\varphi_\alpha$.

In sum, we find that for averaged deformation $e_{ij} = e_{11}\delta_{i1}\delta_{j1}$ $(i, j = 1,2)$., the linear (with respect to e_{11}) deformation of the binder element located between adjacent reinforcing fibers in a given reinforcing layer is equal to

$$e^{\varepsilon}_{ll} = e_{11}\cos^2\varphi_\alpha, \quad e^{\varepsilon}_{ln} = e^{\varepsilon}_{nl} = e_{11}\sin\varphi_\alpha\cos\varphi_\alpha, \quad e^{\varepsilon}_{nn} = e_{11}\sin^2\varphi_\alpha \tag{3}$$

(we note the coincidence of Eqs. (3) with the formulas for transformation of tensors in a rotation of coordinate axes. This considerably simplifies subsequent calculations).

Using solution (3) for the case $e_{ij} = e_{11}\delta_{i1}\delta_{j1}$ $(i, j = 1, 2)$, we easily find the local strains of the binder with specification of an arbitrary averaged strain e_{ij}, $e_{i3} = 0$ $(i = 1, 2)$. In fact, the case of averaged tension–compression in the direction of the Ox_2 axis reduces to the above-examined rotation of coordinate axes through 90°, while the case of averaged shear in the plane Ox_1x_2 reduces to the sum of two axial tension–compressions e_{12} in the coordinate system rotated through 45°. Let us present the final expression for the local strains of the binder between adjacent fibers of a reinforcing layer (to do this, it is convenient to use the note made in regard to Eq. (3)):

$$\begin{aligned} e^{\varepsilon}_{ll} &= e_{11}\cos^2\varphi_\alpha + 2e_{12}\sin\varphi_\alpha\cos\varphi_\alpha + e_{22}\sin^2\varphi_\alpha, \\ e^{\varepsilon}_{ln} = e^{\varepsilon}_{nl} &= (e_{22} - e_{11})\sin\varphi_\alpha\cos\varphi_\alpha + e_{12}(\cos^2\varphi_\alpha - \sin^2\varphi_\alpha), \\ e^{\varepsilon}_{nn} &= e_{11}\sin^2\varphi_\alpha - 2e_{12}\sin\varphi_\alpha\cos\varphi_\alpha + e_{22}\cos^2\varphi_\alpha. \end{aligned} \tag{4}$$

Here and below, the superscript ϵ denotes local stresses and strains.

Interlayer Deformation of the Binder. Now we will examine an element of the binder located between adjacent layers of reinforcing fibers with the indices α and α + 1. Being deformed in a cell of the periodicity Π, the binder element experiences a strain equal to the sum of the strain of a cell of periodicity Π on the average e_{ij}, $e_{i3} = 0$ $(i = 1, 2)$ and the strain due to the relative displacement of the fibers of the adjacent α-th and (α + 1)-st reinforcing layers.

Let us calculate the latter strain. To do this, we examine the deformation of a binder element connecting the fibers of adjacent reinforcing layers (Figs. 2 and 3). We begin with the special case $e_{ij} = e_{11}\delta_{i1}\delta_{j1}$ $(i, j = 1, 2)$. Figure 2 shows the axes of the fibers of the α-th and (α + 1)-st layers before and after deformation. The isolated binder element connects fibers OB and FD at point E (in the undeformed state). Point E is common to fibers OB and FD. Let us see where point E goes on fibers OB and FD during the deformation of a cell of periodicity Π. It is evident that in this case the displacement vector of the points of the fibers OB and FD is parallel to the Ox_1 axis. We find its value at point E for fibers OB and FD. For the fiber FD, we find from the similarity of ΔFEe and ΔFDD' that the displacement of point E

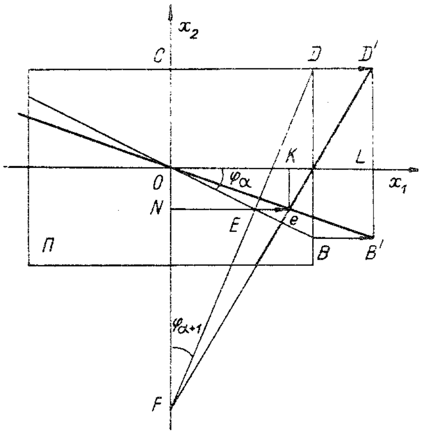

Fig. 2

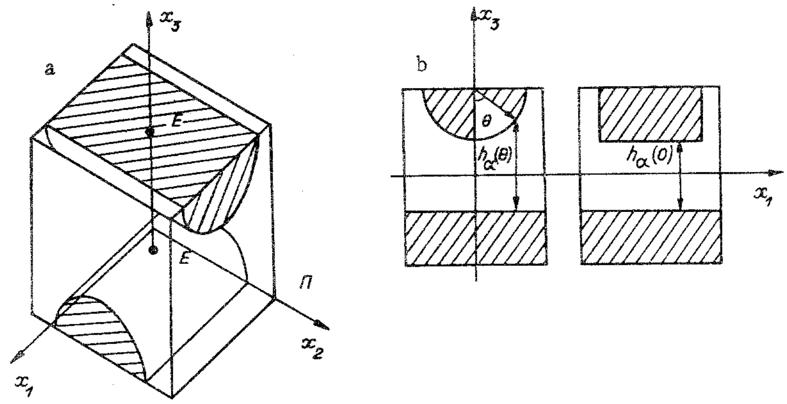

Fig. 3

is $|Ee_{FD}| = |DD'|\frac{|Fe|}{|FD'|}$. Similarly, from the similarity of ΔOEe and ΔOBB′ we find that the displacement of point E is $|Ee_{OB}| = |BB'|\frac{|Oe|}{|OB'|}$.

Let us examine $\triangle OKe, \triangle OLB'$ and $\triangle FNe, \triangle FCD'$. It is not hard to see that $|Fe|/|FD'| = |OK|/|OL| = |Oe|/|OB'|$, from which (since |DD′| = |BB′|) we have $|Ee_{FD}| = |Ee_{OB}|$, i.e. the common point E on fibers OB and FD remains a common point during deformation of the cell П and becomes a certain points. Consequently, there is no mutual displacement of the fibers at the point of their attachment to the element. As a result the binder element lying between the fibers OB and FD undergoes only twisting caused by the rotations of the fibers during deformation. The angle of twist is equal to the change in the angle between the fibers OB and FD:

$$\tau_\alpha = \psi_{\alpha+1} - \psi_\alpha = \delta\psi_\alpha = e_{11}\delta(\sin\varphi_\alpha \cos\varphi_\alpha) \tag{5}$$

(δ is the operator for taking the first difference; $\delta f_\alpha = f_{\alpha+1} - f_\alpha$). In obtaining (5), we used the expression employed above for the angle of rotation of the fibers ψ_α.

Using the solution (5) for the case $e_{ij} = e_{11}\delta_{i1}\delta_{j1}$ (tension–compression along the axis Ox_1), we easily find an expression for the angle of twist of a binder element lying between fibers of adjacent reinforcing layers for an arbitrary strain e_{ij}, $e_{i3} = 0$ ($i = 1, 2$). To do this, it is sufficient to execute the above-noted rotations of the coordinate axes through angles of 90 and 45°. We finally obtain

$$\tau_\alpha = (e_{11} - e_{22})\delta(\sin\varphi_\alpha \cos\varphi_\alpha) + e_{12}\delta(\cos^2\varphi_\alpha - \sin^2\varphi_\alpha). \tag{6}$$

Note 2. The problem of the torsion of the binder element depicted in Fig. 3a is solved only by numerical means. At the same time, it is possible to suggest a simple theoretical scheme for a composite which will make it possible to calculate the maximum interlayer stresses in the binder. The angle of twist of the binder element per unit length does not exceed $\tau_\alpha/h_\alpha(0)$, where $h_\alpha(0)$ is the minimum distance between points of the fibers of the α-th and

(α + 1)-st reinforcing layers (Fig. 3b). The maximum angle of twist per unit length corresponds to the maximum stresses in the twisted element.

We will examine a theoretical scheme for a composite in which the angle of twist of the binder element per unit length is equal to the maximum value. The following condition should be satisfied for this to occur: the kinematics of the framework of reinforcing fibers remains the same, while the distances between the points of adjacent fibers is equal to $h_\alpha(0)$. This condition is satisfied by the composite scheme (Fig. 4), in which the fibers have a rectangular cross section and are described around the corresponding initial fibers. The latter determine all of the geometric dimensions in Fig. 4. In particular, the distances between the fibers of the α-th and (α + 1)-st layers in the theoretical scheme is equal to $h_\alpha(0)$.

Using the scheme described above, we can obtain the binder strength criterion in terms of the averaged stresses or strains of the composite (for the fibers, such a criterion can be obtained on the basis of the results in [1]). Let the material of the binder be described by the strength condition

$$f(\sigma_{ij}^{\varepsilon}) \leqslant \sigma^* \tag{7}$$

(f is a continuous function of $\{\sigma_{ij}^{\varepsilon}\}$). As an example, we point out the widely used quadratic strength condition

$$\sigma_{ij}^{\varepsilon}\sigma_{ij}^{\varepsilon} - \frac{1}{3}(\sigma_{ii}^{\varepsilon})^2 \leqslant \sigma^*. \tag{8}$$

Using Hooke's law for the material of the binder and Eqs. (4), (6), we find that in subjecting the composite to the averaged strain $e_{ij}, e_{i3} = 0\ (i = 1, 2)$, the following local stresses develop in the binder:

interfiber stresses within the α-th layer of reinforcing fibers

$$\begin{gathered}\sigma_{ll}^{\varepsilon} = \frac{E}{1-\nu^2}(e_{ll}^{\varepsilon} + \nu e_{nn}^{\varepsilon}), \quad \sigma_{nn}^{\varepsilon} = \frac{E}{1-\nu^2}(e_{nn}^{\varepsilon} + \nu e_{ll}^{\varepsilon}), \\ \sigma_{ln}^{\varepsilon} = \sigma_{nl}^{\varepsilon} = \frac{E}{2(1+\nu)} e_{ln}^{\varepsilon}, \quad \sigma_{i3} = 0 \text{ with } i = l, n, 3\end{gathered} \tag{9}$$

($\{e_{ij}^{\varepsilon}\}$ are given in (3));

the interlayer stresses

$$\begin{gathered}\sigma_{ij}^{\varepsilon} = \frac{E\nu}{1+\nu}(e_{11} + e_{22} + e_{33})\,\delta_{ij} + \frac{E}{2(1+\nu)} e_{ij}, \\ (i, j) = (1, 1), (1, 2), (2, 1), (2, 2), (3, 3), \\ \sigma_{13}^{\varepsilon} = \sigma_{31}^{\varepsilon} = -\left(x_2 - x_2^E\right)\frac{E}{2(1+\nu)}\frac{\tau_\alpha}{h_\alpha(0)}, \quad \sigma_{23}^{\varepsilon} = \sigma_{32}^{\varepsilon} = \left(x_1 - x_1^E\right)\frac{E}{2(1+\nu)}\frac{\tau_\alpha}{h_\alpha(0)},\end{gathered} \tag{10}$$

where $\left(x_1^E, x_2^E\right)$ are the coordinates of the point E – the axis of rotation of the binder element shown in Figs. 2-4. Equations (10) were obtained on the basis of the theoretical scheme in Fig. 4 and give maximum values of interlaminar stresses.

Having inserted Eqs. (9) and (10) into the binder strength condition, we arrive at the initial binder strength criterion in terms of the averaged strains. Here, insertion of Eqs. (9) and (10) actually leads to strength criteria in the α-th layer of the binder:

the criterion of the strength of the binder with respect to interfiber stresses

$$f_1(\varphi_\alpha, \{e_{ij}\}) \leqslant \sigma^* \tag{11}$$

(the function f_1 is obtained by inserting Eqs. (9) into the binder strength condition (7));

the criterion of the strength of the binder with respect to interlayer stresses

$$f_2(\varphi_\alpha, \varphi_{\alpha+1}, x_1, x_2, \{e_{ij}\}) \leqslant \sigma^* \tag{12}$$

(the function f_2 is obtained by inserting Eqs. (10) into (7)). Equations (11) and (12) explicity state the dependence of the strength conditions both on the averaged strains and on the reinforcement scheme (i.e. on the microstructure of the composite).

Note 3. Using the averaged Hooke's law $\{\sigma_{ij}\} = \{\widehat{a}_{ijkl}\}\{e_{kl}\}$ ($\{\widehat{a}_{ijkl}\}$ is indicated in [1]), we can express the averaged strains $\{e_{ij}\}$ through the averaged stresses: $\{e_{ij}\} = \{H_{ijkl}\}\ \{\sigma_{kl}\}$ ($\{H_{ijkl}\} = \{\widehat{a}_{ijkl}\}^{-1}$). Insertion of the last expression into (11), (12) gives us binder strength

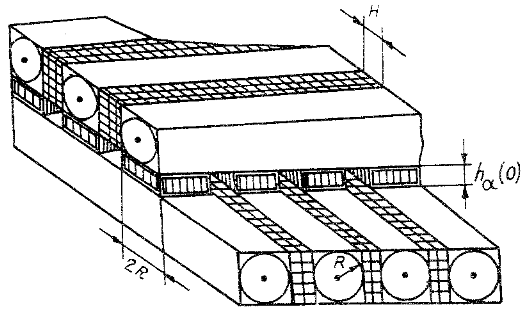

Fig. 4

criterion in terms of the averaged stresses. Allowing for the fact that $\{\hat{a}_{ijkl}\}$ depends on $\varphi_1, \ldots, \varphi_M$ [1], we find the averaged strength criterion in the α-th layer of the binder, corresponding to (11) and (12), in terms of the averaged stresses:

$$F_{1\alpha}(\varphi_1, \ldots, \varphi_M, \{\sigma_{ij}\}) \leqslant \sigma^*, \quad F_{2\alpha}(\varphi_1, \ldots, \varphi_M, x_1, x_2, \{\sigma_{ij}\}) \leqslant \sigma^*. \tag{13}$$

The averaged strength criteria of an arbitrary layer of the binder were obtained above. It would naturally be convenient to have a similar strength criterion for the composite as a whole. With this in mind, we introduce the functionals

$$M(\varphi_1, \ldots, \varphi_M, \{\sigma_{ij}\}) = \max_{\alpha=1,\ldots,M} F_{1\alpha}(\varphi_1, \ldots, \varphi_M, \{\sigma_{ij}\}), \quad N(\varphi_1, \ldots, \varphi_M, \{\sigma_{ij}\}) = \max_{\substack{\alpha=1,\ldots,M \\ (x_1,x_2)}} F_{2\alpha}(\varphi_1, \ldots, \varphi_m, x_1, x_2, \{\sigma_{ij}\}). \tag{14}$$

If $M(\varphi_1, \ldots, \varphi_M, \{\sigma_{ij}\}) \leqslant \sigma^*$, then the first condition in (13) (the strength of the binder in interfiber stresses) is satisfied in all of the layers of binder. If $M(\varphi_1, \ldots, \varphi_M, \{\sigma_{ij}\}) > \sigma^*$, then the inequality $F_{1\alpha*}(\varphi_1, \ldots, \varphi_M, \{\sigma_{ij}\}) > \sigma^*$, is satisfied for a certain index α^*, i.e., the strength condition of the α^*-th binder layer is violated (failure occurs in the layer with the number α^*). The same considerations apply in regard to the functional $N(\varphi_1, \ldots, \varphi_M, \{\sigma_{ij}\})$. It should be noted that in the case of satisfaction of the inequality $N(\varphi_1, \ldots, \varphi_M, \{\sigma_{ij}\}) > \sigma^*$, the conclusion of the failure of a certain layer of binder pertains to the theoretical scheme in Fig. 4.

For composites in general, with allowance for Note 2, the condition $N(\varphi_1, \ldots, \varphi_M, \{\sigma_{ij}\}) \leqslant \sigma^*$ will be sufficient to preserve the integrity of the binder in regard to interlayer stresses. We thus obtain an averaged criterion of binder strength for the composite as a whole:

in regard to interfiber stresses $M(\varphi_1, \ldots, \varphi_M, \{\sigma_{ij}\}) \leqslant \sigma^*$,

in regard to interlayer stresses $N(\varphi_1, \ldots, \varphi_M, \{\sigma_{ij}\}) \leqslant \sigma^*$.

The functionals $m(\varphi_1, \ldots, \varphi_M, \{e_{ij}\})$, $n(\varphi_1, \ldots, \varphi_M, \{e_{ij}\})$, analogous to (14), can be obtained by replacing the functions $F_{1\alpha}(\varphi_1, \ldots, \varphi_M, \{\sigma_{ij}\})$, $F_{2\alpha}(\varphi_1, \ldots, \varphi_M, x_1, x_2, \{\sigma_{ij}\})$ in (14) by the functions $f_1(\varphi_\alpha, \{e_{ij}\})$, $f_2(\varphi_\alpha, \varphi_{\alpha+1}, x_1, x_2, \{e_{ij}\})$ As a result, we have the averaged criterion of binder strength for the composite as a whole in terms of averaged strains:

in regard to interfiber stresses $m(\varphi_1, \ldots, \varphi_M, \{e_{ij}\}) \leqslant \sigma^*$,

in regard to interlayer stresses $n(\varphi_1, \ldots, \varphi_M, \{e_{ij}\}) \leqslant \sigma^*$.

As an example, we obtain the averaged strength criterion of the binder if we assign the strength condition of the binder in the form (8). Let us substitute (9) into (8). After grouping the terms, we find the binder strength criterion relative to interfiber stresses

$$\frac{2}{3}\frac{E^2(1-\nu+\nu^2)}{(1-\nu^2)^2}\left(e_{ll}^{\varepsilon}+e_{nn}^{\varepsilon}\right)^2 - 2\frac{E^2}{(1+\nu)^2}e_{ll}^{\varepsilon}\,e_{nn}^{\varepsilon} + \frac{E^2}{2(1+\nu)^2}e_{ln}^{\varepsilon 2} \leqslant \sigma^*$$

($\{e_{ij}^{\varepsilon}\}$ $(i, j = l, n)$ has obtained in (4)). Making the substitution (4), we have the strength criterion in the α-th layer of binder relative to interfiber stresses

$$\frac{2}{3}\frac{E^2(1-\nu+\nu^2)}{(1-\nu^2)^2}(e_{11}+e_{22})^2-2\frac{E^2}{(1+\nu)^2}\big[(e_{11}^2+e_{22}^2)\sin^2\varphi_\alpha\cos^2\varphi_\alpha+$$
$$+e_{11}e_{22}(\sin^4\varphi_\alpha+\cos^4\varphi_\alpha)+2e_{11}e_{12}\sin\varphi_\alpha\cos\varphi_\alpha(\sin^2\varphi_\alpha-\cos^2\varphi_\alpha)+$$
$$+2e_{22}e_{12}\sin\varphi_\alpha\cos\varphi_\alpha(\cos^2\varphi_\alpha-\sin^2\varphi_\alpha)-4e_{12}^2\sin^2\varphi_\alpha\cos^2\varphi_\alpha\big]+ \quad (15)$$
$$+\frac{E^2}{2(1+\nu)^2}((e_{22}-e_{11})\sin\varphi_\alpha\cos\varphi_\alpha+e_{12}(\cos^2\varphi_\alpha-\sin^2\varphi_\alpha))^2\leqslant\sigma^*.$$

Having taken the maximum of the left side of (15) for $\alpha = 1, \ldots, M$ with fixed $\{e_{ij}\}$ we obtain the averaged strength criterion of the binder relative to interfiber stresses for the composite as a whole.

We similarly find the averaged strength criterion relative to interlaminar stresses for the composite as a whole. Insertion of (10) into (8) leads to the criterion of strength in the α-th layer of binder relative to the interlayer stresses

$$\frac{E^2}{3(1+\nu)(1-2\nu)}(e_{11}+e_{22}+e_{33})^2+\frac{E^2}{4(1+\nu)^2}(e_{11}^2+e_{22}^2+e_{33}^2)+$$
$$+\frac{E^2}{2(1+\nu)^2}[(e_{22}-e_{11})\delta(\sin\varphi_\alpha\cos\varphi_\alpha)+ \quad (16)$$
$$+e_{12}\delta(\cos^2\varphi_\alpha-\sin^2\varphi_\alpha)]^2\frac{(x_1-x_1^E)^2+(x_2-x_2^E)^2}{h_\alpha^2(0)}\leqslant\sigma^*.$$

Having taken the maximum of the left side of (16) for $\alpha = 1, \ldots, M$ and (x_1, x_2) with fixed $\{e_{ij}\}$, we obtain the required result. Having used the averaged Hooke's law [1], we can write the averaged criteria in terms of the averaged stresses.

Let us take a closer look at the strength criterion of the binder for interlaminar stresses (16). The last term in the left side of (16) contains the factor $(x_1-x_1^E)^2+(x_2-x_2^E)^2$. Within the framework of the theoretical scheme in Fig. 4, the largest value of this quantity is $2R^2$ (R is the radius of the reinforcing fibers). The scheme in Fig. 4 gives the maximum values of the stresses which develop in an actual composite. In this connection, satisfaction of (16), with the replacement of $(x_1-x_1^E)^2+(x_2-x_2^E)^2$ by $2R^2$, guarantees that the binder will not fail.

Now we turn our attention to the fact that the combination $[(x_1-x_1^E)^2+(x_2-x_2^E)^2]/h_\alpha^2(0)$ is present in the left side of (16). This combination has the order $R^2/h_\alpha^2(0)$ (R is the radius of the reinforcing fibers, $h_\alpha(0)$ is the distance between fibers in the theoretical scheme in Fig. 4). As a result, for reinforcement scheme characterized by the ratio

$$h_\alpha(0)/R\leqslant 1 \quad (17)$$

(and even more so for $h_\alpha(0)/R \ll 1$), the strength criterion for interlayer stresses can be violated at low levels of external loads. For composites working with a prescribed average stress–strain state, this effect can be neutralized by selecting appropriate reinforcement schemes. The scheme chosen should minimize the modulus of the expression in square brackets in (16). This effect is related to the filler content of the composite. With an increase in the volumetric content of reinforcing fibers, conditions (17) must be satisfied and the strength of the binder decreases.

Note 4. The preceding estimates were made on the basis of the theoretical scheme in Fig. 4. Let us examine a composite reinforced with fibers of circular cross section (Fig. 3b). In this case, the distance between fibers $h_\alpha = h_\alpha(\theta)$ (h_α, θ are shown in Fig. 3b). The angle of twist of a binder element per unit length $\tau_\alpha/h_\alpha(\theta) = \tau_\alpha/(h_\alpha(0) + R(1-\cos\theta))$ (see Fig. 3a). As can be easily verified, its maximum value is equal to $\tau_\alpha\Big/\sqrt{2h_\alpha(0)/R+h_\alpha^2(0)/R^2}$. It follows from this that the above-described effect is seen in this case as well. The volumetric content of fibers at which the effect begins to be manifest is determined from the condition $h_\alpha(0)/R \sim 1$. This corresponds to a volumetric fiber content of about 60%. The maximum possible content of fibers of circular cross section in composites of the above-noted type is about 78%.

Note 5. The problem examined here is characterized by the presence of two parameters: $\epsilon \ll 1$ and $E_f/E \gg 1$. The question of the range of application of a two-scale model was addressed in [1]. It follows from [1] that the parameters ϵ, E_f, and E must be connected by the relation $\varepsilon^2 E_f/E \ll 1$ [1, p. 258], or $\varepsilon \ll \sqrt{E/E_f}$. For composites with soft matrices, $E_f/E \sim 10^2$-10^4 [2]. Then $\epsilon \ll 10^{-1}$-10^{-2}. Satisfaction of these relations is typical of the composites used in practice [2].

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